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Distribution of Fractional Parts and Approximation of Functions with Singularities by Bernstein Polynomials

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1. INTRODUCTION

In this paper we establish an asymptotic property of the sequence of fractional parts $\{n\alpha\}$ $(n = 1, 2, 3, ...)$ and its relation with the asymptotic behavior of the uniform approximation by Bernstein polynomials

$$B_n(f;; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \tag{1}$$

of functions $f(x)$ having a singularity on the interval $0 \leqslant x \leqslant 1$.

As usual, we denote by $[a]$ the greatest integer which does not exceed a and set $\{a\} = a - [a]$, the fractional part of a. It is well known that for every irrational number α the sequence $\{n\alpha\}$ $(n = 1, 2, 3, ...)$ is everywhere dense in $[0, 1]$. By a theorem of H. Weyl [1], the numbers of $\{n\alpha\}$ are distributed uniformly in $[0, 1]$, i.e., for arbitrary $t \in [0, 1]$, we have the limit relation

$$\lim_{n \to \infty} \frac{v_n(t)}{n} = t \tag{2}$$

where $v_n(t)$ denotes the number of values of the sequence $\{k\alpha\}$ $(1 \leqslant k \leqslant n)$ for which $\{k\alpha\} < t$.

These results are, however, false if α is rational. In this case the corresponding sequence is discrete and periodic with period determined by α.

The purpose of this paper is to show that, in spite of this distinctive

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nature of the sequence of fractional parts $\{n\alpha\}$ characterizing the set of irrational numbers, there is an asymptotic property of this sequence valid for both irrational and rational values of α. This property, in particular, turns out to be connected with some problems in the theory of approximation by algebraic polynomials of functions having singularities. This connection permits the establishment of the following result which complements the asymptotic equality (2).

THEOREM 1. *Let* $0<\alpha<1$ *and*

$$\begin{aligned} c_k(\alpha) &= \{k\alpha\}\alpha, && \text{if} \quad \{k\alpha\}<1-\alpha \\ &= (1-\{k\alpha\})(1-\alpha), && \text{if} \quad \{k\alpha\}\geqslant 1-\alpha. \end{aligned} \tag{3}$$

Then the series

$$\sum_{k=1}^{\infty} \frac{c_k(\alpha)}{k^r} \tag{4}$$

converges for all $\alpha\in(0,1)$ *if and only if* $r>1$. *For* $r>1$ *and arbitrary* $\alpha\in[0,1]$, *we have the asymptotic equality*

$$\lim_{n\to\infty} n^{r-1}\cdot\sum_{k=n}^{\infty}\frac{c_k(\alpha)}{k^r}=\frac{\alpha(1-\alpha)}{2(r-1)}. \tag{5}$$

Note that, by (3),

$$c_k(\alpha)\leqslant\alpha(1-\alpha), \qquad c_k(1-\alpha)=c_k(\alpha) \qquad (k=1,2,3,...).$$

Hence, for $t\geqslant\alpha(1-\alpha)$, *all the values* $c_k(\alpha)$ *satisfy the condition* $c_k(\alpha)\leqslant t$. *On the other hand, if* $0<t<\alpha(1-\alpha)$, *where* α *is irrational and* $\tilde{v}_n(t)$ *is the number of values* $c_k(\alpha)$ $(1\leqslant k\leqslant n)$ *for which* $c_k(\alpha)<t$, *then by* (2), *we have the asymptotic equality*

$$\tilde{v}_n(t)=\frac{nt}{\alpha(1-\alpha)}+\sigma(n), \qquad \text{as } n\to\infty,$$

which completes the limit relation (5) *for Theorem* 1.

2. SHARPENING OF THEOREM 1 FOR α RATIONAL

In case when α is rational Theorem 1 can be proved directly by using the following elementary lemma which may be useful in other problems as well.

LEMMA. *For every rational number* $\alpha = p/q \in (0, 1)$ *with* $(p, q) = 1$, *we have*

$$\frac{1}{q} \sum_{k=1}^{q} c_k(\alpha) = \frac{1}{2} \alpha(1 - \alpha). \tag{6}$$

Proof of the Lemma. Let $M^{(q)}$ denote the set of positive integers $k \leqslant q$ for which $\{k\alpha\} < 1 - \alpha$ and $N^{(q)}$ the set of positive integers $k \leqslant q$ for which $\{k\alpha\} \geqslant 1 - \alpha$. We show that for $\alpha = p/q \in (0, 1)$, $(p, q) = 1$, and for any $k \in M^{(p)}$ with $k \neq q$, there is a unique $S \in M^{(q)}$ such that

$$\{k\alpha\} + \{s\alpha\} = 1 - \alpha. \tag{7}$$

Indeed, consider the Diophantine equation

$$px - qy = q - p - q\{k\alpha\}, \tag{8}$$

whose right-hand side is positive if $k \in M^{(q)}$. Since $(p, q) = 1$, there exist integral solutions. All solutions of (8) can be written in the form

$$x = qm - k - 1,$$

$$y = pm - [k\alpha] - 1,$$

where m is an arbitrary integer. If $k \in M^{(q)}$ then, among these solutions, $x \in M^{(q)}$ if and only if $m = 1$. Consequently, for every $k \in M^{(q)}$, $k \neq q$, there is a unique $s = q - k - 1 \in M^{(q)}$ which satisfies (7). In case when $k = q$ the unique value s which satisfies this condition is contained in $N^{(q)}$ and is equal to $q - 1$ $(m = 2)$.

Analogously, we can show that for every $k \in N^{(q)}$ with $k \neq q - 1$, there is a unique $S \in N^{(q)}$ such that

$$\{k\alpha\} + \{s\alpha\} = 2 - \alpha. \tag{9}$$

Here, instead of (8), we consider the Diophantine equation

$$px - qy = 2q - p - q\{ka\}$$

whose general solution is

$$x = qm - k - 1,$$

$$y = pm - [k\alpha] - 2.$$

If $k \in N^{(q)}$ then, among these solutions, $x \in N^{(q)}$ if and only if $m = 1$. Hence, for every $k \in N^{(q)}$, $k \neq q - 1$, there is a unique $s = q - k - 1 \in N^{(q)}$ which satisfies (9). In case when $k = q - 1$ the unique value s which satisfies this condition is contained in $M^{(q)}$ and is equal to $q (m = 2)$.

Thus, for $\alpha = p/q \in (0, 1)$, $(p, q) = 1$, it follows from (3) that

$$\sum_{k=1}^{q} c_k(\alpha) = \sum_{k \in M^{(q)}} c_k(\alpha) + \sum_{k \in N^{(q)}} c_k(\alpha)$$
$$= \alpha \sum_{k \in M^{(q)}} \{k\alpha\} + (1-\alpha) \sum_{k \in N^{(q)}} (1 - \{k\alpha\}) = \frac{1}{2} q\alpha(1-\alpha),$$

and we obtain (6).

By (6), for rational $\alpha = p/q \in (0, 1)$, $(p, q) = 1$, and for any positive integer μ, we have the inequalities

$$\frac{\alpha(1-\alpha)}{2(\mu+1)^r q^{r-1}} \leqslant \sum_{k=\mu q+1}^{(\mu+1)q} \frac{c_k(\alpha)}{k^r} \leqslant \frac{\alpha(1-\alpha)}{2\mu^r q^{r-1}} \qquad (r > 0).$$

Therefore, if $n = mq + v$ $(m = 1, 2, 3, \ldots; 0 \leqslant v \leqslant q)$ and $r > 1$ then

$$\sum_{k=n}^{\infty} \frac{c_k(\alpha)}{k^r} = \sum_{k=n}^{(m+1)q} \frac{c_k(\alpha)}{k^r} + \sum_{\mu=m+1}^{\infty} \sum_{k=\mu q+1}^{(\mu+1)q} \frac{c_k(\alpha)}{k^r}$$
$$= \sum_{\mu=m+1}^{\infty} \sum_{k=\mu q+1}^{(\mu+1)q} \frac{c_k(\alpha)}{k^r} + \mathcal{O}\left(\frac{1}{n^r}\right) = \frac{\alpha(1-\alpha)}{2(r-1)n^{r-1}} + \mathcal{O}\left(\frac{1}{n^r}\right),$$

i.e., we have (5) and, for $r = 1$, the series (4) is not convergent.

For α irrational, the proof of Theorem 1 requires different arguments and can be carried out by using the theory of approximation of functions by algebraic polynomials.

3. Approximation of Functions Having Singularities by Bernstein Polynomials and Fractional Parts

The study of the asymptotic behavior of the uniform polynomial approximation of functions whose derivatives have a jump discontinuity in [0, 1] reduces to the study of uniform approximation of the simplest type of such functions $f(x) = |x - \alpha|$ $(0 < \alpha < 1)$. If we approximate f by Bernstein polynomials (1) then, due to the convexity of this function the maximal value of uniform approximation is achieved at the point $x = \alpha$, i.e.,

$$\max_{0 \leqslant x \leqslant 1} |f(x) - B_n(f; x)| = B_n(f; \alpha) \qquad (n = 1, 2, 3, \ldots). \tag{10}$$

The asymptotic behavior of the sequence $B_n(f; \alpha)$ for $f(x) = |x - \alpha|$ $(0 < \alpha < 1)$ was studied by R. Bojanic [2], who established the limit relation

$$\lim_{n \to \infty} \sqrt{n} \cdot B_n(f; \alpha) = \sqrt{\frac{2\alpha(1-\alpha)}{\pi}}. \tag{11}$$

An earlier method which was applied in [3] to the function $f(x) = x^\alpha$ allows to obtain a different formula for the left-hand side of (11). This makes it possible to obtain a relation between the sequence (10) and the sequence of fractional parts $\{n\alpha\}$ $(n = 1, 2, 3...)$.

THEOREM 2. *If $f(x) = |x - \alpha|$ $(0 < \alpha < 1)$ then*

$$B_n(f; \alpha) = \sqrt{\frac{2}{\pi\alpha(1-\alpha)}} \cdot \sum_{k=n}^{\infty} \frac{c_k(\alpha)}{k^{3/2}} + \mathcal{O}\left(\frac{1}{n}\right),$$

uniformly for $n \geq 1$ and $\alpha \in [\varepsilon, 1 - \varepsilon]$, where $\varepsilon > 0$ is arbitrary.

Proof of Theorem 2. Write the sequence $B_n(f; \alpha)$ in the form

$$B_n(f; \alpha) = \sum_{k=n}^{\infty} (B_k(f; \alpha) - B_{k+1}(f; \alpha))$$

and consider the differences $B_k(f; \alpha) - B_{k+1}(f; \alpha)$ $(k \geq n)$. From (1) it follows immediately that

$$\begin{aligned} &B_k(f; \alpha) - B_{k+1}(f; \alpha) \\ &= \sum_{v=1}^{k} \left(\frac{k+1-v}{k+1} f\left(\frac{v}{k}\right) - f\left(\frac{v}{k+1}\right)\right. \\ &\left.+ \frac{v}{k+1} f\left(\frac{v-1}{k}\right)\right) \binom{k+1}{v} \alpha^v (1-\alpha)^{k+1-v}. \end{aligned} \tag{12}$$

Note that if $0 < \alpha < 1$ and $m = [k\alpha] + 1$ then we always have

$$\frac{m-1}{k} \leq \alpha, \qquad \frac{m}{k} > \alpha.$$

Thus, for the function $f(x) = |x - \alpha|$, it follows from (12) that we have

$$\begin{aligned} &B_k(f; \alpha) - B_{k+1}(f; \alpha) \\ &= \left(\frac{k+1-m}{k+1} f\left(\frac{m}{k}\right) - f\left(\frac{m}{k+1}\right)\right. \\ &\left.+ \frac{m}{k+1} f\left(\frac{m-1}{k}\right)\right) \binom{k+1}{m} \alpha^m (1-\alpha)^{k+1-m}. \end{aligned}$$

Consider the two possible locations of the point $m/(k+1)$: $m/(k+1) > \alpha$ and $m/(k+1) \leq \alpha$. Let M_n be the set of positive integers $k \geq n$ for which

$m/(k+1) > \alpha$ and N_n the set of positive integers $k \geqslant n$ for which $m/(k+1) \leqslant \alpha$. If $k \in M_n$ then we have for the function $f(x) = |x - \alpha|$:

$$\frac{k+1-m}{k+1} f\left(\frac{m}{k}\right) - f\left(\frac{m}{k+1}\right) + \frac{m}{k+1} f\left(\frac{m-1}{k}\right) = 2\frac{m}{k+1}\left(\alpha - \frac{m-1}{k}\right)$$
$$= 2\frac{[k\alpha]+1}{k+1}\frac{\{k\alpha\}}{k} = 2\alpha\frac{\{k\alpha\}}{k} + 2\frac{(1-\alpha-\{k\alpha\})\,\{k\alpha\}}{k(k+1)}.$$

By the local limit theorem of probability theory, for $0 < \alpha < 1$, we have the asymptotic relation

$$\binom{k+1}{m}\alpha^m(1-\alpha)^{k+1-m} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{k\alpha(1-\alpha)}} + \mathcal{O}\left(\frac{1}{k}\right), \tag{13}$$

uniformly with respect to k and $\alpha \in [\varepsilon, 1-\varepsilon]$ $(\varepsilon > 0)$. Hence, in our case when $k \in M_n$, $f(x) = |x - \alpha|$, we have

$$B_k(f;\alpha) - B_{k+1}(f;\alpha) = \sqrt{\frac{2\alpha}{\pi(1-\alpha)}} \cdot \frac{\{k\alpha\}}{k^{3/2}} + \mathcal{O}\left(\frac{1}{k^2}\right).$$

If $k \in N_n$ we have

$$\frac{k+1-m}{k+1} f\left(\frac{m}{k}\right) - f\left(\frac{m}{k+1}\right) + \frac{m}{k+1} f\left(\frac{m-1}{k}\right)$$
$$= 2\left(1 - \frac{m}{k+1}\right)\left(\frac{m}{k} - \alpha\right)$$
$$= 2\left(1 - \alpha + \frac{\{k\alpha\} - 1 + \alpha}{k+1}\right)\frac{1 - \{k\alpha\}}{k}$$
$$= 2(1-\alpha)\frac{1-\{k\alpha\}}{k} + 2\frac{(\{k\alpha\} - 1 + \alpha)\,(1 - \{k\alpha\})}{k(k+1)}.$$

Hence, by (13), for $k \in N_n$, we have

$$B_k(f;\alpha) - B_{k+1}(f;\alpha) = \sqrt{\frac{2(1-\alpha)}{\pi\alpha}} \cdot \frac{1-\{k\alpha\}}{k^{3/2}} + \mathcal{O}\left(\frac{1}{k^2}\right).$$

Therefore, we obtain

$$B_n(f;\alpha) = \sqrt{\frac{2\alpha}{\pi(1-\alpha)}} \sum_{k \in M_n} \frac{\{k\alpha\}}{k^{3/2}} + \sqrt{\frac{2(1-\alpha)}{\pi\alpha}} \sum_{k \in N_n} \frac{1-\{k\alpha\}}{k^{3/2}} + \mathcal{O}\left(\frac{1}{n}\right),$$

or equivalently,

$$B_n(f;\alpha)=\sqrt{\frac{2}{\pi\alpha(1-\alpha)}}\sum_{k=n}^{\infty}\frac{c_k(\alpha)}{k^{3/2}}+\mathcal{O}\left(\frac{1}{n}\right).$$

4. Return to Theorem 1

Validity of asymptotic equality (5), for $r=3/2$, follows from the comparison of the statement of Theorem 2 with the result of R. Bojanic mentioned earlier. For the other values of r, the proof of Theorem 1 can be obtained by the means of the Abel transformation. If $r=3/2+\rho$ and

$$\sigma_n=\sum_{k=n}^{\infty}\frac{c_k(\alpha)}{k^{3/2}} \qquad (n=1,2,3,...)$$

then

$$\sum_{k=n}^{\infty}\frac{c_k(\alpha)}{k^r}=\sum_{k=n}^{\infty}\frac{1}{k^\rho}(\sigma_k-\sigma_{k+1})=\frac{\sigma_n}{n^\rho}-\sum_{k=n}^{\infty}\sigma_{k+1}\cdot\Delta\frac{1}{k^\rho},$$

where

$$\Delta\frac{1}{k^\rho}=k^{-\rho}-(k+1)^{-\rho}.$$

On the other hand,

$$\Delta\frac{1}{k^\rho}=\frac{\rho}{(k+\theta_k)^{\rho+1}} \qquad (0<\theta_k<1),$$

and we already proved that

$$\sigma_n=\frac{\alpha(1-\alpha)}{n^{1/2}}+\mathcal{O}\left(\frac{1}{n^{1/2}}\right).$$

Consequently, for $r\leqslant 1$, $0<\alpha<1$, the series (4) does not converge and, for $r>1$,

$$\sum_{k=n}^{\infty}\frac{c_k(\alpha)}{k^r}=\frac{\alpha(1-\alpha)}{n^{r-1}}-\alpha(1-\alpha)\rho\sum_{k=n}^{\infty}\frac{1}{(k+1)^{1/2}(k+\theta_k)^{\rho+1}}+\mathcal{O}\left(\frac{1}{n^{r-1}}\right).$$

It follows that, for $r>1$, we have

$$\sum_{k=n}^{\infty}\frac{c_k(\alpha)}{k^r}=\frac{\alpha(1-\alpha)}{n^{r-1}}+\alpha(1-\alpha)\rho\sum_{k=n}^{\infty}\frac{1}{k^r}+\mathcal{O}\left(\frac{1}{n^{r-1}}\right),$$

and we obtain the limit relation (5).

Note that, by using the terminology of probability theory, we can reformulate the statement of Theorem 1 as follows:

For every real number $\alpha \in (0, 1)$, the mathematical expected value $E_n(\alpha)$ of the discrete random variable taking value $c_k(\alpha)$ $(k = 1, 2, 3,...)$ with probability

$$\begin{aligned} p_k^{(n)} &= 0, & 0 < k < n \\ &= \left(k^r \sum_{\nu=n}^{\infty} \frac{1}{\nu^r}\right)^{-1}, & k \geqslant n, \end{aligned}$$

for arbitrary $r > 1$, satisfies the limit relation

$$\lim_{n \to \infty} E_n(\alpha) = \tfrac{1}{2}\alpha(1 - \alpha).$$

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